Maths for Computing Tutorial 5

Note: Using strong induction for problems from 1 to 6 may make proof simpler.

1. Let $a_0 = 0$, and $a_n = a_0 + a_1 + a_2 + \ldots + a_{n-1} + n$, if $n \ge 1$. Prove that for all non-negative integers n, $a_n = 2^n - 1$.

2. Prove that every amount of postage of 12 rupees or more can be formed using Rs. 4 and Rs. 5 stamps.

3. Suppose you begin with a pile of $n \ge 2$ stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals n(n - 1)/2.

4. Prove that in the game of Chomp if the 2nd player does not have a winning strategy then the 1st player must have.

5. Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the equal number of matches initially, then the second player can always guarantee a win.

6. Prove that for all positive integers $n \ge 2$, it is possible to organise a round-robin tournament of *n* football teams in

a) n - 1 rounds if n is even,

b) *n* rounds if *n* is odd.

A round is a set of games in which each team plays one opponent if n is even, and there is only one idle team if n is odd. A round-robin tournament is a tournament in which any pair of teams meet exactly once. Use strong induction.

7. Suppose we want to prove that $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$ for all positive integers *n*.

a) Try to prove this inequality using mathematical induction. If you are stuck in the inductive step, see b).

b) Show that mathematical induction can be used to prove the stronger inequality $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$ for all integers greater than 1, which, together with a

verification of the original inequality for n = 1, establishes the weaker inequality we originally tried to prove using mathematical induction.

8. Prove that the union of countably many countable sets is countable.

9. Prove that the set of real numbers containing no other digits apart from a finite number of 1s in their decimal representation is countable. (Numbers allowed in the sets are 1.11,1.1,.1,111, etc. Numbers not allowed in the set are1.11...,1.01,1.2,1.311, etc.)

10. Prove that if *A* is countable but *B* is not, then $B \setminus A$ is uncountable.

Solutions

Solution 1

Let P(n) be $a_n = 2^n - 1$.

Basis Step: For n = 0,1, the statement is trivially true as $a_0 = 0$, $a_1 = 1$. **Inductive Step:** Suppose the statement is true for all P(0), P(1), P(2),..., P(k). We will prove P(k + 1) now. We know that $a_{k+1} = a_0 + a_1 + a_2 + \ldots + a_k + k$. From inductive hypothesis, we can say that $a_i = 2^i - 1$, $\forall i \in [0,k]$. Then,

$$\begin{aligned} a_{k+1} &= a_0 + a_1 + a_2 + \ldots + a_k + k \\ &= (2^0 - 1) + (2^1 - 1) + \ldots + (2^k - 1) + k \\ &= 2^0 + 2^1 + \ldots + 2^k - 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Solution 2

Let P(n) be the statement that postage of *n* rupees can be formed using Rs. 4 and Rs. 5 stamps. **Basis Step:** P(12), P(13), P(14), P(15) are true because postage of 12, 13, 14, 15 rupees can be formed with 3 Rs. 4 stamps, 2 Rs. 4 stamps + 1 Rs. 5 stamps, 1 Rs. 4 stamps + 2 Rs. 5 stamps, 3 Rs. 5 stamps.

Inductive Step: Assume P(12), P(13), P(14), ..., P(k) are true and prove that P(k + 1) is true, for any arbitrary $k \ge 15$.

Using IH, we can say that postage for (k - 3) rupees can be formed as $k - 3 \ge 12$. Therefore, postage of k + 1 rupees can be formed by a Rs. 4 stamps and the stamps we used to form postage of k - 3 rupees.

Solution 3

We use strong induction to prove it.

Basis Step: For n = 2, the statement is trivially true as for n = 2 the only split will result in 1 which is equal to 2(2 - 1)/2.

Inductive Step: Assuming P(1), P(2), ..., P(k - 1), we will prove P(k).

Suppose we split a pile of k stones into r size and k - r size piles. Then, the sum of products will be r(k - r) + sum of products when you further break r size and k - r size piles. From inductive hypothesis, we know that for r size and k - r size piles, the sum of the product will always be r(r - 1)/2 and (k - r)(k - r - 1)/2. Therefore, the sum of the products while breaking k size pile will be r(k - r) + r(r - 1)/2 + (k - r)(k - r - 1)/2, which can be simplified to k(k - 1)/2.

Solution 4

We divide every position in the game of chomp into two categories:

- 1. P-positions: The previous player, one who has just moved, has a winning strategy.
- 2. *N*-positions: The next player, one who is about to move, has a winning strategy.

We will prove now that all positions can be either a P-position or an N-position.

Basis Step: When only one cookie is remaining, it is clearly a P-position.

Inductive Step: Assume that all positions with k or lesser cookies can be either a P-position or an N-position. Let us consider a position with k + 1 cookie. There are two cases possible. *Case 1*: It is possible to move to a P-position by picking a cookie. Then, it is an N-position as the next player will simply move to that P-position.

Case 2: It is not possible to move to a P-position by picking a cookie. That means every move will lead to an N-position. If that is the case then position with k + 1 cookies is a P-position. The winning strategy for previous player is to simply wait and let next player play and then use the winning strategy for that position.

Now, in the beginning of a Chomp game, player 2 can be thought of as the previous player and player 1 as the next player.

Solution 5

We will use strong induction to prove that 2nd player can always guarantee a win. **Basis Step:** When both piles contain 1 match each, 2nd player is guaranteed to win as 1st player can pick only one pile after which the 2nd player can pick the other remaining pile. **Inductive Step:** Assume 2nd player can guarantee his win when both piles contain 1,2,..., or k matches. We prove now that 2nd player can guarantee his win when both piles contain k + 1 matches. If 1st player picks entire k + 1 matches from a pile, then 2nd player can pick the other pile and win the game. Otherwise, if 1st player picks l matches from one pile, then player 2 will also pick l matches from the other pile, reducing the game to a game of (k + 1 - l, k + 1 - l) matches where player 1 has to move first. From IH, in such a game player 2 can guarantee a win.

Solution 6

We will use strong induction.

Basis Step: For n = 2, the tournament will have just 1 round where team 1 will play with 2. For n = 3, 1st round: 1 - 2,3, 2nd round: 2 - 3,1, 3rd round: 1 - 3,2. For n = 4, 1st round: 1 - 2,3 - 4, 2nd round: 1 - 3,2 - 4, 3rd round: 1 - 4,2 - 3. **Inductive Step:** Assume we can organise a round-robin tournament for 1,2,...,k teams. We will prove now that we can organise the round-robin for k + 1 teams.

Case 1: k + 1 is even.

First divide the teams into two equal sets of size k' = (k + 1)/2. Let teams in first set be $a_1, a_2, ..., a_{k'}$ and teams in second set be $b_1, b_2, ..., b_{k'}$. Suppose k' is even. Then from IH we can organise a round-robin tournament of k' - 1 rounds for each of both sets of teams. To construct a

round-robin tournament for k + 1 teams we can start by clubbing the pair of rounds of these two round-robin tournaments. After the clubbing, every team in first set has played with every team in the first set, and the same is true for the second set teams. Now, we need to let every team of first set play with every team of second set. In the first round, we can have the following matches: $(a_1, b_1), (a_2, b_2), \dots, (a_{k'}, b_{k'})$. In the second round, we can have the following matches: $(a_1, b_2), (a_2, b_3), \dots, (a_{k'}, b_1)$. In the third round, we can have the following matches: $(a_1, b_3), (a_2, b_4), \dots, (a_{k'}, b_2)$, and so on. These extra rounds will be k' many. In total, the number of rounds will be k' - 1 + k' = k.

If k' is odd, then from IH we can organise a round-robin tournament of k' rounds for each of both sets of teams such that in every round one team is sitting idle. It is easy to see that one team will be sitting idle in exactly one round. Again to construct a round-robin tournament for k + 1 teams we can start by clubbing the pair of rounds of these two round-robin tournaments. But this time apart from usual clubbing we let the idle teams play with each other as well. Let idle teams of first set in every round be $x_1, x_2, \ldots, x_{k'}$ and idle teams of second set in every round be $y_1, y_2, \ldots, y_{k'}$. Since every x_i has already played with y_i , now we need to let every x_i play with y_js such that $i \neq j$. In the first round, we can have the following matches: $(x_1, y_2), (x_2, y_3), \ldots, (x_{k'}, y_1)$. In the second round, we can have the following matches: $(x_1, y_3), (x_2, y_4), \ldots, (x_{k'}, y_2)$. In the third round, we can have the following matches: $(x_1, y_3), (x_2, y_4), \ldots, (x_{k'}, y_2)$. In the third round, we can have the following matches: $(x_1, y_4), (x_2, y_5), \ldots, (x_{k'}, y_3)$, and so on. These extra rounds will be k' - 1 many. In total, the number of rounds will be k' + k' - 1 = k.

Case 2: k + 1 is odd

We can add an extra team to the set of k + 1 teams. For k + 2 teams, we can again prove that we can organise a round-robin tournament in k + 1 rounds using the above argument. (Note that using above argument involves using inductive hypothesis which is not an issue as $(k + 2)/2 \le k$.) From the round-robin tournament of k + 1 rounds we can simply eliminate the extra added team to get a round-robin tournament of k + 1 rounds for odd k + 1 teams.

Solution 7

a) Let's try to prove the inequality using induction, **Basis Step:** For n = 1, $1/2 < 1/\sqrt{3}$. Hence, trivially true. **Inductive Step:** We assume now that $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2k-1}{2k} < \frac{1}{\sqrt{3k}}$ and prove that $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2k+1}{2k} < \frac{1}{\sqrt{3k+3}}$.

Let multiply on both sides of our inductive hypothesis by $\frac{2k+1}{2k+2}$. We will get $\frac{1}{2k+2} = \frac{3}{2k-1} + \frac{2k+1}{2k+2} = \frac{1}{2k+1} + \frac{2k+1}{2k+2}$.

$$\overline{2} \cdot \overline{4} \cdot \dots \cdot \overline{2k} \cdot \overline{2k+2} < \overline{\sqrt{3k}} \cdot \overline{2k+2}$$

If we can prove $\frac{1}{\sqrt{3k}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+3}}$, then we will be done. Unfortunately, this inequality is

not true.

b) Let's try to prove a stronger inequality, i.e., $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$, for $n \ge 2$, that

implies the original inequality. Together with the fact that original inequality is true for n = 1, it completes the proof.

Basis Step: For n = 2, $3/8 < 1/\sqrt{7}$. Hence, trivially true. **Inductive Step:** We assume now that $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2k-1}{2k} < \frac{1}{\sqrt{3k+1}}$ and prove that $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$.

Let multiply on both sides of our inductive hypothesis by $\frac{2k+1}{2k+2}$. We will get $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}$

If we can prove $\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$, then we will be done. We can assume that this

inequality is true and simplify it to 3 < 4 in a reversible fashion. Hence the

 $\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$ is true and inductive step is complete.

Solution 8

Suppose we have countably infinite many sets such that each set is also countably infinite. Let us name these sets as A_1 , A_2 , A_3 , etc, and arrange the elements of these sets in an infinite 2d matrix such that first row contains elements of A_1 , second row contains elements of A_2 , third row contains elements of A_3 , and so on. Now we can get a list of $A_1 \cup A_2 \cup A_3 \cup \ldots$ by traversing the elements of this 2d matrix in a dovetail order and dropping the duplicate elements.

If the number of sets are finite or sets themselves are finite then we can add dummy sets and dummy elements to have countably infinite sets where every set is also countably infinite. After listing the elements using the above procedure we can drop duplicate elements, the dummy element and elements of dummy sets.

Solution 9

We can arrange all the real numbers containing only finitely many 1s in a matrix. In the *i*th row of this matrix, we have, in increasing order, real numbers that have i - 1 many 1s before decimal.

Any real number that contains *i* many 1s before decimal and total *j* many 1s after decimal will be present in the (i + 1)th row and (j + 1)th column. Now we can simply traverse all the numbers in the matrix in a dovetail order to create a sequence of real numbers that contains only finitely many 1s. Since we have a sequence of such real numbers, their set is countable.

Solution 10

It is a proof by contradiction. Suppose *A* is countable, *B* is uncountable, and $B \setminus A$ is countable. Then, $A \cup (B \setminus A) = A \cup B$ is also countable. Since *B* is a subset of a countable set $A \cup B$, *B* is also countable. But this is a contradiction as we assumed *B* is uncountable.